

SCALAR FIELD COSMOLOGY — TOWARD DESCRIPTION OF DYNAMIC COMPLEXITY OF COSMOLOGICAL EVOLUTION

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We study the dynamical evolution of cosmological models with the Robertson-Walker symmetry with a scalar field non-minimally coupled to gravity and barotropic matter. For this aim we use dynamical system methods. We have found a type of evolutionary path which links between all important events during the evolution, the cosmological singularity of finite time, inflation, radiation and matter dominating epoch and the accelerated phase expansion of the universe. We point out importance of finding the new generic solution called a twister solution for a deeper description of the evolution of the Universe. We demonstrate that including the non-minimal coupling leads to a new, richer evolutionary cosmological scenario in comparison to the case of minimal coupling.

1 Introduction

The standard method of description matter content in cosmology bases on the concept of perfect fluid approximation, where pressure and energy density satisfy the equation of state. Different cosmological epochs constitute solutions of the Einstein equations corresponding different forms of the equation of state, usually postulated in a linear form with respect to the energy density. On the other hand if we study very early stages evolution of the Universe then characterization of matter content in terms of the barotropic equation of state is not adequate. In the quantum epoch including matter in the form of scalar field with the potential seems to be more suitable. We propose to describe matter in the form of both barotropic matter and single scalar field with the potential. We also assumed that there is present a nonzero coupling constant scalar field to the gravity.

A non-minimal coupling appeared naturally in quantum theory of the scalar field as generated by quantum corrections or required by renormalization of the theory.¹ The value of this coupling constant becomes important for cosmology. Recently, it has been constructed an extended model of inflation with a non-minimal coupling between the inflaton field and the Ricci scalar curvature.^{2,3,4,5} It was shown in particular that the non-minimal inflation can be realized within the Standard Model (SM) or minimal extension of it.⁶ In the SM the Higgs boson inflation and dark matter is considered by Clark et al.⁷ The non-minimal coupling plays also an important role in the context of description of quintessence epoch.^{8,9,10}

The main aim of this paper is to present a generic solution of the non-minimal coupling cosmology called the twister solution because of the shape of its trajectory in the phase space. We explore here methods of dynamical systems because their advantage of representing all solutions for all admissible initial conditions. The phase space is organized by critical points which represent stationary solutions for which right-hand sides of the dynamical system vanish and trajectories joining them represent the evolution of the system. A new type of evolution appears only if the coupling constant is different from minimal and conformal. This solution leads naturally to the quintessence epoch through the twister

solution. We characterize properties of this acceleration by calculation of so called state-finder parameters.

We assume the spatially flat FRW universe filled with the non-minimally coupled scalar field and barotropic fluid with the equation of the state coefficient w_m . The action is

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left[\frac{1}{\kappa^2} R - \varepsilon \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \xi R \phi^2 \right) - 2U(\phi) \right] + S_m, \quad (1)$$

where $\kappa^2 = 8\pi G$, $\varepsilon = +1, -1$ corresponds to canonical and phantom scalar fields, respectively, the metric signature is $(-, +, +, +)$, $R = 6 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right)$ is the Ricci scalar, a is the scale factor and a dot denotes differentiation with respect to the cosmological time t and $U(\phi)$ is the scalar field potential function. S_m is the action for the barotropic matter part.

The dynamical equation for the scalar field we can obtain from the variation $\delta S / \delta \phi = 0$

$$\ddot{\phi} + 3H\dot{\phi} + \xi R\phi + \varepsilon U'(\phi) = 0, \quad (2)$$

and energy conservation condition from the variation $\delta S / \delta g^{\mu\nu} = 0$

$$\mathcal{E} = \varepsilon \frac{1}{2} \dot{\phi}^2 + \varepsilon 3\xi H^2 \phi^2 + \varepsilon 3\xi H(\phi^2) + U(\phi) + \rho_m - \frac{3}{\kappa^2} H^2. \quad (3)$$

Then the Einstein equation for the flat FRW model and the conservation condition read

$$\frac{3}{\kappa^2} H^2 = \rho_\phi + \rho_m, \quad \text{and} \quad \dot{H} = -\frac{\kappa^2}{2} \left[(\rho_\phi + p_\phi) + \rho_m(1 + w_m) \right] \quad (4)$$

where the energy density and the pressure of the scalar field are

$$\rho_\phi = \varepsilon \frac{1}{2} \dot{\phi}^2 + U(\phi) + \varepsilon 3\xi H^2 \phi^2 + \varepsilon 3\xi H(\phi^2), \quad (5)$$

$$p_\phi = \varepsilon \frac{1}{2} (1 - 4\xi) \dot{\phi}^2 - U(\phi) + \varepsilon \xi H(\phi^2) - \varepsilon 2\xi (1 - 6\xi) \dot{H} \phi^2 - \varepsilon 3\xi (1 - 8\xi) H^2 \phi^2 + 2\xi \phi U'(\phi). \quad (6)$$

2 Non-minimal scalar field cosmology as a dynamical system

In what follows we introduce the energy phase space variables $x \equiv \frac{\kappa \dot{\phi}}{\sqrt{6}H}$, $y \equiv \frac{\kappa \sqrt{U(\phi)}}{\sqrt{3}H}$, $z \equiv \frac{\kappa}{\sqrt{6}} \phi$, which are suggested by the conservation condition

$$\frac{\kappa^2}{3H^2} \rho_\phi + \frac{\kappa^2}{3H^2} \rho_m = \Omega_\phi + \Omega_m = 1. \quad (7)$$

The acceleration equation can be rewritten to the form

$$\dot{H} = -\frac{\kappa^2}{2} (\rho_{\text{eff}} + p_{\text{eff}}) = -\frac{3}{2} H^2 (1 + w_{\text{eff}}) \quad (8)$$

where the effective equation of the state parameter reads

$$w_{\text{eff}} = \frac{1}{1 - \varepsilon 6\xi (1 - 6\xi) z^2} \left[-1 + \varepsilon (1 - 6\xi) (1 - w_m) x^2 + \varepsilon 2\xi (1 - 3w_m) (x + z)^2 + (1 + w_m) (1 - y^2) - \varepsilon 2\xi (1 - 6\xi) z^2 - 2\xi \lambda y^2 z \right] \quad (9)$$

where $\lambda = -\frac{\sqrt{6}}{\kappa} \frac{1}{U(\phi)} \frac{dU(\phi)}{d\phi}$.

The dynamics of the model can be written down as the 4-dimensional autonomous dynamical system in variables x , y , z and λ where differentiation is with respect to time τ defined as $\frac{d}{d\tau} = \left[1 - \varepsilon 6\xi (1 - 6\xi) z^2 \right] \frac{d}{d \ln a}$.¹⁰ However, when the function $\Gamma = \frac{d^2 U(\phi)}{d\phi^2} U(\phi) \left(\frac{dU(\phi)}{d\phi} \right)^{-2}$ is assumed to be the function of λ then function $z = z(\lambda)$ can be determined from $z(\lambda) = -\int [\lambda^2 (\Gamma(\lambda) - 1)]^{-1} d\lambda$ and the system can be

further reduced to the form of the 3-dimensional dynamical system

$$\begin{aligned} x' = & -(x - \varepsilon \frac{1}{2} \lambda y^2) \left[1 - \varepsilon 6\xi(1 - 6\xi)z(\lambda)^2 \right] + \frac{3}{2} (x + 6\xi z(\lambda)) \left[-\frac{4}{3} - 2\xi \lambda y^2 z(\lambda) \right. \\ & \left. + \varepsilon(1 - 6\xi)(1 - w_m)x^2 + \varepsilon 2\xi(1 - 3w_m)(x + z(\lambda))^2 + (1 + w_m)(1 - y^2) \right], \end{aligned} \quad (10)$$

$$\begin{aligned} y' = & y \left(2 - \frac{1}{2} \lambda x \right) \left[1 - \varepsilon 6\xi(1 - 6\xi)z(\lambda)^2 \right] + \frac{3}{2} y \left[-\frac{4}{3} - 2\xi \lambda y^2 z(\lambda) \right. \\ & \left. + \varepsilon(1 - 6\xi)(1 - w_m)x^2 + \varepsilon 2\xi(1 - 3w_m)(x + z(\lambda))^2 + (1 + w_m)(1 - y^2) \right], \end{aligned} \quad (11)$$

$$\lambda' = -\lambda^2 (\Gamma(\lambda) - 1) x \left[1 - \varepsilon 6\xi(1 - 6\xi)z(\lambda)^2 \right] \quad (12)$$

if we postulate the exact form of Γ , for example $\Gamma(\lambda) = 1 - \frac{1}{\lambda^2}(\alpha + \beta\lambda + \gamma\lambda^2)$. Note that this discussion is not restricted to the specific potential function but is generic in the sense that it is valid for any function $\Gamma(\lambda)$ for which $z(\lambda)$ exists. In general there are various potential functions commonly used in the literature of the subject.¹⁰

The qualitative analysis of differential equations allows to study all properties of the solutions without solving the dynamical system. First, we calculate the critical points which correspond mathematically vanishing right-hand sides of the system and physically stationary states. Then we linearize the system around these points. The information about the character of critical point and their stability is contained in eigenvalues of the linearization matrix. If all real parts of eigenvalues are negative (positive) then critical point is stable (unstable). If the all eigenvalues are real of different signs then we have critical point of saddle type organized through the stable and unstable submanifolds. It is interesting that for our system all important events during the cosmic evolution are represented by these critical points which traces a generic trajectory representing the evolution of the universe. Let us enumerate all these points (see also¹⁰).

- 1) Finite scale factor singularity; eigenvalues are $l_1 = 6\xi$, $l_2 = 6\xi$, $l_3 = 12\xi$, and critical point are an unstable node for positive coupling $\xi > 0$ and a stable node for negative coupling $\xi < 0$.
- 2a) Fast-roll inflation; $l_1 = 0$, $l_2 = 12\xi$, $l_3 = -12\xi$, and the critical point is non-hyperbolic.
- 2b) Slow-roll inflation; $l_1 = l_2 = l_3 = 0$ and the critical point is degenerated.
- 3) Radiation domination epoch generated by the non-minimal coupling; there are two critical points. For the phantom scalar field and $\xi > 0$ eigenvalues are $l_1 = 0$, $l_2 > 0$, $l_3 < 0$. For the canonical scalar field and $\xi > 0$ eigenvalues are $l_1 = 6\xi(1 - 3w_m)$, $l_2 = 12\xi$, $l_3 = -6\xi$.
- 4) Matter domination epoch; $l_{1,2} = -\frac{3}{4}[(1 - w_m) \pm [(1 - w_m)^2 - \frac{16}{3}\xi(1 - 3w_m)]^{1/2}]$, $l_3 = \frac{3}{2}(1 + w_m)$, and the critical points are non-degenerated for $w_m \neq -1$ and $w_m \neq \frac{1}{3}$.
- 5) The present accelerated expansion epoch.

In the most general case without assuming any specific form of the potential function we are unable to find coordinates of this point. In spite of this we are able to formulate general conditions for stability of this critical point. From the Routh-Hurwitz criterion we have that the following conditions should be fulfilled to assure stability of this critical point

$$Re[l_{1,2,3}] < 0 \iff 3\xi \frac{h'(\lambda_5^*)}{z'(\lambda_5^*)} (y_5^*)^2 > 0 \quad \text{where} \quad h(\lambda) = \lambda z(\lambda)^2 + 4z(\lambda) - \frac{\lambda}{\varepsilon 6\xi}. \quad (13)$$

To emphasize the acceleration epoch in the twister solution, the state-finder parameters are calculated: $q = -\ddot{a}/aH^2$ is the deceleration parameter and $r = \ddot{a}/aH^3$ (see Fig. 1).

3 Conclusions

In this paper we pointed out the presence of the new interesting solution for the non-minimally coupled scalar field cosmology which we called the twister solution (because of the shape of the corresponding trajectory in the phase space). This type of the solution is very interesting because in the phase space it represents the 3-dimensional trajectory which interpolates different stages of evolution of the universe, namely, the radiation dominated, dust filled and accelerating universe. We are able to find linearized solutions around all these intermediate phases, and hence, parameterizations for $w_{\text{eff}}(a)$ in different epochs of the universe history. It is interesting that the presented structure of the phase space is allowed only

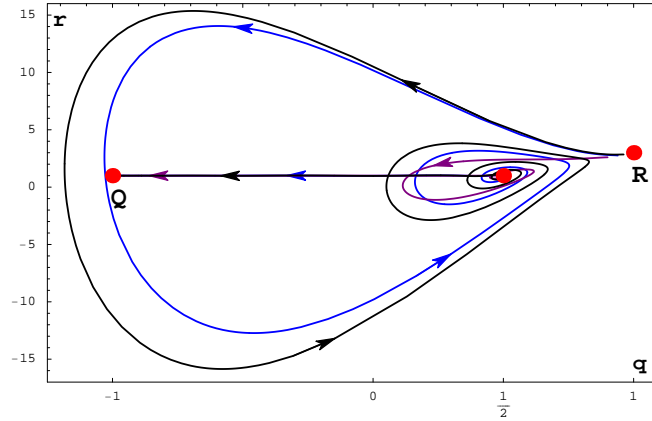


Figure 1: The state-finder parameter diagnostic for the twister quintessence scenario and $z(\lambda) = \frac{\lambda}{\alpha}$. The trajectories (black $\xi = 6$, $\alpha = -6$, blue $\xi = 4$, $\alpha = -4$ and purple $\xi = 2$, $\alpha = -2$) represent a twister type solution which interpolates between R – the radiation dominated universe (a saddle type critical point), the matter dominated universe (an unstable focus critical point) and Q – the accelerating universe (a stable critical point).

for non-zero value of the coupling constant, therefore it is a specific feature of the non-minimally coupled scalar field cosmology. Our general conclusion is that in the description of dynamical complexity the cosmic evolution including both barotropic matter and non-minimally scalar field leads to a new richer dynamics.

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